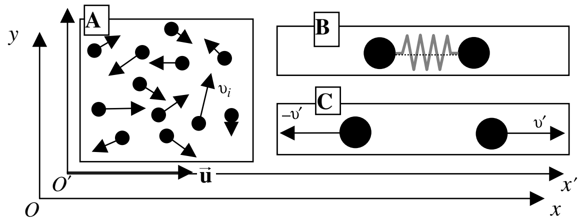

Figure 1. Two model systems storing kinetic and potential energies. Box A contains non-interacting particles of equal masses. Box B contains a compressed spring squeezed between two bodies of equal masses connected with a thread. The masses of the thread and the spring are negligibly small. When the thread is cut and the spring is released, the two masses move freely (box C). All the boxes are at rest in the reference frame O'. The frame O' moves with velocity **u** relative to the frame O.

$\mathbf{F} \parallel \mathbf{u}$, treating **F** and **u** as scalars is a legitimate simplification and the velocity change is given by

$$\mathrm{d}u = \frac{u + a'\,\mathrm{d}t'}{1 + \frac{ua'\,\mathrm{d}t'}{c^2}} - u \approx a'\,\mathrm{d}t'\left(1 - \frac{u^2}{c^2}\right). \tag{3}$$

Using the *time dilation formula* $\mathrm{d}t = \gamma\,\mathrm{d}t'$ we calculate that the change of the velocity u equals $\mathrm{d}u = a'(1 - u^2/c^2)^{3/2}\,\mathrm{d}t$. Integration of this differential equation is straightforward and the result reads $a't = u(1 - u^2/c^2)^{-1/2}$. Therefore $f(\mathbf{u}) = \mathbf{u}(1 - u^2/c^2)^{-1/2}$ and equation (2) follows. Working backwards, we can also see that from equation (2) and $\mathbf{u} \parallel \mathbf{F}$ follows $\mathbf{F}' = \mathbf{F}$. Similar to [24], the presented derivation does not use the momentum conservation law explicitly.

The kinetic energy can be obtained in a standard way using the definition of the energy as the work done by an external force when the particle moves from position 1 to position 2:

$$\varepsilon_{\mathrm{kin}} = \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\mathrm{d}P}{\mathrm{d}t}\,\mathrm{d}x = \int_0^{p_2} u\,\mathrm{d}P. \tag{4}$$

After integration [5–8], equation (1) follows.

3. Mass–energy relation

Now we consider a box filled with non-interacting particles (ideal gas) having equal masses m (figure 1, box A). In a reference frame O' where the box is at rest the velocities of the particles are given by vectors $\boldsymbol{\upsilon}'_i \equiv (\upsilon'_{i,x}, \upsilon'_{i,y}, \upsilon'_{i,z})$ $(i = 1, \ldots, N)$. We also select a reference frame O relative to which the box moves along the x-axis with velocity **u**. The velocities of all particles in the box are now υ_i $(i = 1, \ldots, N)$ and we can write their total momentum as

$$\mathbf{P} = \sum_{i=1}^{N} \frac{m\upsilon_i}{\sqrt{1 - \upsilon_i^2/c^2}}. \tag{5}$$

Using the relativistic transformation for all the velocities we obtain for the x-component of the total momentum

$$P_x = \sum_{i=1}^{N} \frac{m\,\frac{\upsilon'_{i,x}+u}{1+\upsilon'_{i,x}u/c^2}}{\sqrt{1 - \frac{(\upsilon'_{i,x}+u)^2+(\upsilon'^2_{i,y}+\upsilon'^2_{i,z})(1-u^2/c^2)}{c^2(1+\upsilon'_{i,x}u/c^2)^2}}}. \tag{6}$$

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Explicit derivation of the relativistic mass–energy relation for internal kinetic and potential energies of a composite system

T Plakhotnik

School of Physical Sciences, University of Queensland, St Lucia, QLD 4072, Australia

E-mail: taras@physics.uq.edu.au

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Abstract
A straightforward derivation of relativistic expressions for the mechanical momentum, kinetic and total energies, and mass–energy equivalence (including potential energy) which does not require any knowledge of the energy–momentum relation for electromagnetic waves or consideration of elastic collisions, but is directly based on Newton's second law and Lorentz's transformations, is presented in this paper. The existence of an invariant force is shown to be important for the validity of the relativistic mechanics.

1. Introduction

The mass–energy equivalence is probably the most famous equation of the twentieth century science. The simplicity of $E = mc^2$ is such that the equation is recognized even by non-physicists and de facto became a trademark of modern physics. However, many elementary text books [1–8] fail to present a valid derivation of this equation. They frequently simply state the final result [1–4] or, for example, using the expression for the relativistic kinetic energy

$$\varepsilon_{\text{kin}} = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2 \tag{1}$$

argue [5–8] that the kinetic energy is the difference between a total relativistic energy and a rest energy and therefore the form of equation (1) validates mc^2 as the rest energy. This argument is rather confusing and does not prove anything because the way in which the expression for the kinetic energy is split into velocity-dependent and velocity-independent parts is arbitrary. An approach based on Einstein's type thought experiments [9–13] involves some preliminary knowledge about momentum and energy of electromagnetic fields. This approach demonstrates that there is a mass change related to emission or absorption of a freely propagating electromagnetic pulse by a massive body but Einstein's original statement [10], "The fact that the energy withdrawn from the body becomes energy of radiation evidently makes no difference . . . " does not look so evident. It implies that all forms of energy are equivalent or at least can be reversibly and with 100% efficiency transformed into the electromagnetic energy of a freely propagating electromagnetic pulse. Why not assume that only a propagating electromagnetic pulse carries an equivalent mass? Ironically, nuclear fission, a process most frequently used to illustrate mass–energy equivalence, does not release exclusively electromagnetic radiation. This problem is most intensively discussed in relation to the potential energy [14, 15] and is perhaps a reason why more advanced texts [16–20] derive the energy–mass equivalence by considering in detail elastic and inelastic collisions between two particles in different reference frames and/or by invoking symmetry considerations [21]. When such a derivation involves arguments based on relations between 4-vectors [17, 18, 22], it becomes quite elegant but, in fact, the properties of 4-vectors should be included as postulates in the relativistic theory. An interesting and purely mechanical analogy of Einstein's derivation [23] is a rare exception from the mainstream.

2. Relativistic momentum and energy

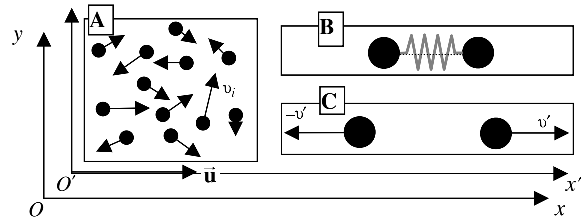
In this paper, we present not only a straightforward derivation of the mass–energy relation but also simple derivations of equation (1) and a well-known expression for the relativistic momentum

$$\mathbf{P} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{2}$$

solely based on Newton's laws of classical mechanics and the relativistic Lorentz's transformations. Such an approach also substantially demystifies the famous $E = mc^2$.

The starting point of the reasoning is Newton's law for a particle $\mathrm{d}\mathbf{P}/\mathrm{d}t = \mathbf{F}$ which, if $\mathbf{F}$ is constant and the particle is at rest at $t = 0$ in a stationary lab frame *XYZ*, can be reduced to $\mathbf{P} = \mathbf{F}t$. The momentum $\mathbf{P}$ is proportional to the inertial mass of the particle and is a function of its velocity $\mathbf{P} = mf(\mathbf{u})$. Hence $f(\mathbf{u}) = \mathbf{F}t/m$. Now we consider a *non-relativistic* acceleration $a' \equiv \mathbf{F}'/m$ of this particle at every given instant in a reference frame $X'Y'Z'$ moving relative to XYZ with a velocity equal to the instantaneous velocity of the particle. Because $\mathbf{F}$ is a constant, it follows that $\mathbf{u} \parallel \mathbf{F} \parallel \mathbf{F}'$ and generally speaking one can assume that $\mathbf{F} = g(u)\mathbf{F}'$, where $g(u)$ is a function of the particle's speed. As an example, consider a Lorentz force between a charged probe-particle moving in a normal direction towards a uniformly charged plane. Because of the symmetry of the problem, a non-zero component of any field acting on the particle should be normal to the plane and hence parallel to the velocity of the probe. Therefore the force is purely electrostatic and is independent of the distance from the plane and particle's speed because there is no Lorenz contraction in the direction perpendicular to the particle's velocity and consequently there is no change in the charge density of the plane. Thus we obtain $\mathbf{F}' = \mathbf{F}$ and as a consequence $\mathbf{P} \equiv mf(\mathbf{u}) = ma't$.

Let the particle be at rest at time t' in a reference frame $X'Y'Z'$. At later moments new moving frames should be chosen to match the velocity of the accelerating particle but for an infinitely short time interval $\mathrm{d}t'$ the classical Newton's law holds and the particle velocity in the frame $X'Y'Z'$ increases from zero to $a'\,\mathrm{d}t'$. In the reference frame *XYZ*, the change of the velocity of the particle $\mathrm{d}\mathbf{u}$ can be obtained using the *Lorentz velocity transformation*. Because



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After integration [5–8], equation (1) follows.

3. Mass–energy relation

Now we consider a box filled with non-interacting particles (ideal gas) having equal masses m (figure 1, box A). In a reference frame O' where the box is at rest the velocities of the particles are given by vectors $\boldsymbol{v}_i' \equiv (v_{i,x}', v_{i,y}', v_{i,z}')$ $(i = 1, \ldots, N)$. We also select a reference frame O relative to which the box moves along the x-axis with velocity $\mathbf{u}$. The velocities of all particles in the box are now v_i $(i = 1, \ldots, N)$ and we can write their total momentum as

$$\mathbf{P} = \sum_{i=1}^{N} \frac{m v_i}{\sqrt{1 - v_i^2/c^2}}. \tag{5}$$

Using the relativistic transformation for all the velocities we obtain for the x-component of the total momentum

$$P_x = \sum_{i=1}^{N} \frac{m\,\frac{v_{i,x}' + u}{1 + v_{i,x}' u/c^2}}{\sqrt{1 - \frac{(v_{i,x}'+u)^2 + (v_{i,y}'^2 + v_{i,z}'^2)(1 - u^2/c^2)}{c^2(1 + v_{i,x}' u/c^2)^2}}}. \tag{6}$$

After simple algebra this can be rewritten as

$$P_x = \sum_{i=1}^{N} \frac{m(\upsilon'_{i,x} + u)}{\sqrt{(1 - u^2/c^2)\left(1 - \upsilon_i'^2/c^2\right)}}, \tag{7}$$

where $\upsilon_i'^2 \equiv \upsilon_{i,x}'^2 + \upsilon_{i,y}'^2 + \upsilon_{i,z}'^2$. Because the particles move chaotically, summation in (7) includes terms with opposite but equal in magnitude velocities $\upsilon'_{i,x}$, therefore

$$P_x = \sum_{i=1}^{N} \frac{mu}{\sqrt{(1 - u^2/c^2)\left(1 - \upsilon_i'^2/c^2\right)}}. \tag{8}$$

Similar consideration results in $P_y = P_z = 0$, which also follows from the symmetry of the problem. On the other hand, considering the box with the particles inside as a single object, we expect that the general equation (2) with yet unknown mass M holds. Comparing equation (2) with m replaced by M and equation (8), we obtain the equality

$$M = \sum_{i=1}^{N} \frac{m}{\sqrt{1 - \upsilon_i'^2/c^2}} = \sum_{i=1}^{N} \left(m + \frac{\varepsilon_{\mathrm{kin},i}}{c^2}\right) \tag{9}$$

where $\varepsilon_{\mathrm{kin},i}$ is the kinetic energy of the ith particle. The second equality in (9) follows from equation (1). Thus, we arrive at the conclusion that the rest mass of the box with moving particles is equal to the mass of the particles at rest plus the total kinetic energy of the particles (that is the internal energy of an ideal gas) divided by c^2. Note that although the middle part of equation (9) can be viewed as a sum of 'relativistic masses' of the gas particles, the concept of a relativistic mass can be misleading [25] and is not used in our derivation.

To treat potential energy, we consider two particles and a compressed spring which is placed between them (figure 1, box B). The mass of the spring is assumed to be zero. When the spring is released, the two particles start to move in the frame O' in opposite directions with equal speeds. This mechanistic model represents decay of an unstable system (nuclear, for example). When the spring is completely released, the two particles do not interact anymore and can be treated like the ideal gas in the previous consideration. Therefore the total momentum of two such unbound particles (figure 1, box C) in the reference frame O is given by equation (8) with $N = 2$. Because at any time, the two particles (bound or unbound) do not interact with external world (they together represent a closed system), the total momentum of the bounded particles should also be given by equation (8) (momentum conservation principle) where υ'_1 and υ'_2 are the velocities of the particles if the spring were completely released. Hence, even for the bounded system equation (9) holds if one understands that $\varepsilon_{\mathrm{kin},1}$ and $\varepsilon_{\mathrm{kin},2}$ are the kinetic energies of the particles after all potential energy of the spring is transformed into the particle energies. Therefore, the total kinetic energy in equation (9) can be replaced by the stored potential energy of the bound system (energy conservation principle) assuming that the potential energy is zero when the particles are separated by a large distance. This concludes the derivation of the mass–energy equivalence.

4. Conclusion

Looking back, one clearly sees that relativistic mechanics follow from Newtonian mechanics, Lorentz transformations and the assumption about the existence of a frame invariant force. The calculations are not really challenging and because the *inertial* mass is just a *proportionality factor* in the expression for the momentum the mass energy relation does not look too mysterious (for an interesting discussion of the mass–energy equivalence, see [26]). The relation between the inertial mass and gravity is not a subject of the special theory of relativity and is simply postulated in general relativity.

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